MATH 579 Exam 8 Solutions

1. Let a_n denote the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no two white squares are adjacent.

   The set of permitted colorings can be divided into two categories. Those that end in a blue or red square start with any permitted coloring of length $n-1$. Those that end in a white square start with any permitted coloring of length $n-2$, then either a red or blue square, then the white square. This gives the recurrence relation $a_n = 2a_{n-1} + 2a_{n-2}$. By inspection, $a_0 = 1, a_1 = 3, a_2 = 8$. The characteristic polynomial is $r^2 - 2r - 2$, which has roots $r_1 = 1 + \sqrt{3}$ and $r_2 = 1 - \sqrt{3}$; hence the general solution is $a_n = \alpha(r_1)^n + \beta(r_2)^n$. Applying the initial conditions we get $1 = a_0 = \alpha + \beta, 3 = a_1 = \alpha r_1 + \beta r_2$, which has solution $\alpha = (3 + 2\sqrt{3})/6, \beta = (3 - 2\sqrt{3})/6$. Hence the solution is $a_n = \frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n$.

2. Solve the recurrence $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + n \ \ (n \geq 2)$.

   The homogeneous relation $a_n = a_{n-1} + 2a_{n-2}$ has characteristic polynomial $r^2 - r - 2 = (r-2)(r+1)$; hence the general hom. solution is $a_n = \alpha 2^n + \beta(-1)^n$. We guess a linear polynomial $a_n = tn + s$ for the nonhom. $tn + s = t(n-1) + s + 2(t(n-2) + s) + n = n(t + 2t + 1) + (-t + s - 4t + 2s)$. Hence $t = 3t + 1, s = -5t + 3s$, which has solution $t = -1/2, s = -5/4$, hence $a_n = (-2n-5)/4$ is a specific solution and $a_n = \alpha 2^n + \beta(-1)^n + (-2n-5)/4$ is the general solution. Applying the initial conditions, $0 = a_0 = \alpha + \beta - 1.25$, $0 = a_1 = 2\alpha - \beta - 1.75$, which has solution $\alpha = 1, \beta = 1/4$. Hence the solution is $a_n = (2^{n+2} + (-1)^n - 2n - 5)/4$

3. Codewords from the alphabet $\{0, 1, 2, 3\}$ are called *legitimate* if they have an even number of 0's. How many legitimate codewords are there, of length k?

   If there are a_k legitimate codewords of length k, there are $4^k - a_k$ illegitimate codewords of length k. Legitimate codewords of length k come in two types: those that end in $1, 2, 3$ begin with a legitimate codeword of length $k-1$, those that end in 0 begin with an illegitimate codeword of length $k-1$. Hence $a_k = 3a_{k-1} + (4^{k-1} - a_{k-1}) = 2a_{k-1} + 4^{k-1}$, with initial condition is $a_1 = 3$. The homogeneous equation has solution $a_k = \alpha 2^k$, and we guess $a_k = \beta 4^k$ for the nonhomogeneous version. $\beta 4^k = 2\beta 4^{k-1} + 4^{k-1}$, which has solution $\beta = 1/2$. We now apply $a_1 = 3$ to the general solution $a_k = \alpha 2^k + 4^k/2$, to get $3 = a_1 = \alpha 2 + 4/2$, which has solution $\alpha = 1/2$. Hence the solution is $a_k = (2^k + 4^k)/2$.

4. Prove that $a_n = \alpha 2^n + \beta$ is the general solution to $a_n = 3a_{n-1} - 2a_{n-2}$. In other words, prove that for all possible initial conditions (a_0, a_1), there is exactly one (α, β) that gives the recurrence (relation+initial conditions).

   Both 2^n and 1 are solutions to the recurrence relation (either by plugging in, or by considering the characteristic polynomial). Hence every (α, β) will give a_n that solves the recurrence relation; the only remaining issue is the initial conditions. The system $a_0 = \alpha + \beta, a_1 = 2\alpha + \beta$ has one and only one solution, for any (a_0, a_1), namely $\alpha = a_1 - a_0, \beta = 2a_0 - a_1$.

5. Solve the recurrence $a_1 = 2, na_n + na_{n-1} - a_{n-1} = 2^n \quad (n \geq 2)$. HINT: Set $b_n = na_n$.

   Using the hint we get the recurrence $b_n = -b_{n-1} + 2^n$, with initial condition $b_1 = 2$ (note that $b_{n-1} = (n-1)a_{n-1} = na_{n-1} - a_{n-1}$). This has homogeneous solution $b_n = \alpha(-1)^n$, and we guess $b_n = \beta 2^n$ for the non-homogeneous solution. $\beta 2^n = -\beta 2^{n-1} + 2^n$, which has solution $\beta = 2/3$. Hence the general solution is $b_n = \alpha(-1)^n + 2^{n+1}/3$. Applying the initial condition we get $2 = b_1 = \alpha(-1) + 4/3$, which has solution $\alpha = -2/3$. Hence $b_n = \frac{-2(-1)^n + 2^{n+1}}{3}$ and $a_n = \frac{-2(-1)^n + 2^{n+1}}{3n}$.

6. How many ways are there to place five (identical) nonattacking rooks on a 5×5 chessboard, with no rooks occupying places $(1,1), (2,2), (3,3), (4,4)$?
   Note: the forbidden squares are four of the five diagonal squares.

   There must be one rook on each row. Let P_i (for $1 \leq i \leq 4$) denote the property that the rook on row i is on the diagonal (column i), and let A_i denote the set of placements that have property P_i. With no restrictions, there are 5! placements (5 choices for rook on row 1, then 4 choices for rook on row 2, etc.). $|A_1| = 4!$, since after placing the rook in the first row in its required place, there are four choices for the next rook, three for the following, etc. Similarly, $|A_2| = |A_3| = |A_4| = 4!$. Hence $\sum |A_i| = \binom{4}{1} 4!$. $|A_1 \cap A_2| = 3!$, since after placing the two rooks perforce, we place the three remaining rooks. Hence $\sum |A_i \cap A_j| = \binom{4}{2} 3!$. Continuing similarly, we have $\binom{4}{0} 5! - \binom{4}{1} 4! + \binom{4}{2} 3! - \binom{4}{3} 2! + \binom{4}{4} 1! = 53$.

Exam results: High score=105, Median score=73, Low score=50 (before any extra credit)

Current exam averages: $\underbrace{90, 88, 87, 84,}_{\text{might get on my door}}$ 83, 82, 78, 74, 73, 73, 73, 72, 66, 64, 63, 63, 61, 47

These averages were computed with exam 9 dropped (worst case scenario).

With wishful thinking, let's assume everyone will score 100 on Exam 9 and on the final.
Optimistic course grades: 96, 94, 93, 92, 91, 90, 89, 88, 87, 86, 86, 85, 85, 82, 81, 80, 79, 71